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From Bayesianism to the Epistemic View of Mathematics†

RICHARD JEFFREY. ***Subjective Probability: The Real Thing.*** Cambridge: Cambridge University Press, 2004. ISBN 0-521-82971-2 (hbk), 0-521-53668-5 (pbk). Pp. xvi + 124.

Reviewed by JON WILLIAMSON*

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Subjective Probability: The Real Thing is the last book written by the late Richard Jeffrey, a key proponent of the Bayesian interpretation of probability.

Bayesians hold that probability is a mental notion: saying that the probability of rain is 0.7 is just saying that you believe it will rain to degree 0.7. Degrees of belief are themselves cashed out in terms of bets—in this case you consider 7:3 to be fair odds for a bet on rain. There are two extreme Bayesian positions. *Strict subjectivists* think that an agent can adopt whatever degrees of belief she likes, as long as they satisfy the axioms of probability. Thus your degree of belief in rain and degree of belief in no rain must sum to one but are otherwise unconstrained. At the other extreme, *objectivists* claim that an agent's background knowledge considerably narrows down the choice of appropriate degrees of belief. In particular, if you know only that the frequency of rain is 0.7 then you should believe it will rain to degree 0.7; if you know absolutely nothing about the weather then you should set your degree of belief in rain to be 0.5; in neither of these cases is there room for subjective choice of degree of belief. In this book, Jeffrey advocates what is sometimes called *empirically-based subjectivism*, a position that lies between the two extremes of strict subjectivism and objectivism. According to this position, knowledge of frequencies constrains degree of belief, but lack of knowledge does not impose any constraints, so that if you know nothing about the weather you may adopt any degree of belief in rain you like.[1]

The aim of the book is not so much to justify this point of view as to provide a comprehensive exposition of probability theory from the perspective that it offers. The book succeeds admirably: Jeffrey presents a broad range of standard topics concerning Bayesianism, including the

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* Department of Philosophy, University of Kent, Canterbury CT2 7NF, U.K. j.williamson@kent.ac.uk

[1] See [Williamson, 2005b] for a more detailed introduction to this spectrum of positions.

betting interpretation of degrees of belief, a discussion of objective chance, the application of Bayesianism to scientific reasoning, conditionalisation, expectation, exchangeability, and decision theory.

Naturally much of the discussion of these topics focuses on Jeffrey's own multifarious contributions to the subject. For example, Jeffrey devotes much more space to his own version of conditionalisation than to the more standard Bayesian conditionalisation; he discusses only the version of decision theory that he developed together with Ethan Bolker. The exposition is also terse: the reader would certainly benefit from prior knowledge of the interpretations of probability as well as knowledge of probability theory itself. For these reasons the book is more useful to the expert for its portrayal of Jeffrey's own position than to the beginner wanting an even-handed introduction to the subject. Jeffrey's book would sit well on a shelf between [de Finetti, 1970] and [Jaynes, 2003], idiosyncratic developments of probability theory from the strict subjective and objective perspectives respectively.

Rather than sift through the details of Jeffrey's proposals—it is testament to the strength of Jeffrey's ideas that many of them have become widely accepted and now seem uncontroversial—I shall instead try to outline some ways in which the philosophy of probability can suggest new directions to the philosophy of mathematics in general.

One potential interaction concerns the existence of mathematical entities. Philosophers of probability tackle the question of the existence of probabilities within the context of an interpretation. Questions like 'what are probabilities?' and 'where are they?' receive different answers according to the interpretation of probability under consideration. There is little dispute that axioms of probability admit of more than one interpretation: Bayesians argue convincingly that rational degrees of belief satisfy the axioms of probability; frequentists argue convincingly that limiting relative frequencies satisfy the axioms (except the axiom of countable additivity). The debate is not so much about finding *the* interpretation of probability, but about which interpretation is *best* for particular applications of probability—applications as diverse as those in statistics, number theory, machine learning, epistemology, and the philosophy of science. Now according to the Bayesian interpretation probabilities are mental entities; according to frequency theories they are features of collections of physical outcomes; and according to propensity theories they are features of physical experimental set-ups or of single-case events. So we see that an interpretation is required before one can answer questions about existence. The uninterpreted mathematics of probability is treated in an *if-then*-ist way: if the axioms hold then Bayes's theorem holds; degrees of rational belief satisfy the axioms; therefore degrees of rational belief satisfy Bayes's theorem.

The question thus arises as to whether it may *in general* be most productive to ask what mathematical entities are *within the context of an interpretation*. It may make more sense to ask 'what kind of thing is a Hilbert space in the epistemic interpretation of quantum mechanics?' than 'what kind of thing is a Hilbert space?'. In mathematics it is crucial to ask questions at the right level of generality; so too in the philosophy of mathematics.

Such a shift in focus from abstraction towards interpretation introduces important challenges. For example, the act of interpretation is rarely a straightforward matter—it typically requires some sort of idealisation. While elegance plays a leading role in the selection of mathematics, the world is rather more messy, and any mapping between the two needs a certain leeway. Thus rational degrees of belief are idealised as real numbers, even though an agent would be irrational to worry about the $10^{10^{10}}$th decimal place of her degree of belief. Frequencies are construed as limits of finite relative frequencies, even though that limit is never actually reached. When assessing an interpretation, the suitability of its associated idealisations are of paramount importance. If it makes a *substantial* difference what the $10^{10^{10}}$th decimal place of a degree of belief is, then so much the worse for the Bayesian interpretation of probability. Similarly when interpreting arithmetic or set theory: if it matters that a large collection of objects is not in fact denumerable then one should not treat it as the domain of an interpretation of Peano arithmetic; if it matters that the collection is not in fact an object distinct from its members then one should not treat it as a set. A first challenge, then, is to elucidate the role of idealisation in interpretations.

A second challenge is to demarcate the interpretations that do bestow existence on mathematical entities from those that do not. While some interpretations construe mathematical entities as worldly things, some construe mathematical entities in terms of other uninterpreted mathematical entities. To take a simple example, one may appeal to affine transformations to interpret the axioms of group theory. In order to construe this group as existing, one must go on to say something about the existence of the transformations: one needs a chain of interpretations that is grounded in worldly things. In the absence of such grounding the interpretation fails to impart existence. These interpretations within mathematics are rather different from the interpretations that are grounded in our messy world, in that they tend not to involve idealisation: the transformations really do form a group. But of course the line between world and mathematics can be rather blurry, especially in disciplines like theoretical physics: are quantum fields part of the world, or do they require further interpretation?[2]

[2] [Corfield, 2003, Part IV] discusses interpretations within mathematics.

This shift in focus from abstraction to interpretation is ontological, but not epistemological. That mathematical entities must be interpreted to exist does not mean that uninterpreted mathematics does not qualify as knowledge. Taking an *if-then*-ist view of uninterpreted mathematics, knowledge is accrued if one knows that the consequent does indeed follow from the antecedent, and the role of proof is of course crucial here.[3]

But there is undoubtedly more to mathematics than a collection of if-then statements and a further analogy with Bayesianism suggests a more sophisticated philosophy. Under the Bayesian view probabilities are rational degrees of belief, a feature of an agent's epistemic state; they do not exist independently of agents. According to objective Bayesianism probabilities are also objective, in the sense that two agents with the same background information have little or no room for disagreement as to the probabilities. This objectivity is a result of the fact that an agent's degrees of belief are so heavily constrained by the extent and limitations of her empirical knowledge.

Perhaps mathematics is also purely epistemic, yet objective. Just as Bayesianism considers probabilistic beliefs to be a type of belief—point-valued degrees of belief—rather than beliefs about agent-independent probabilities, mathematical beliefs may also be a type of belief, rather than beliefs about uninterpreted mathematical entities. Just as probabilistic beliefs are heavily constrained, so too mathematical beliefs are heavily constrained. Perhaps so heavily constrained that mathematics turns out to be fully objective, or nearly fully objective (there may be room for subjective disagreement about some principles, such as the continuum hypothesis).

The constraints on mathematical beliefs are the bread and butter of mathematics. Foremost, of course, mathematical beliefs need to be useful. They need to generate good predictions and explanations, both when applied to the real world, *i.e.*, to interpreted mathematical entities, and when applied within mathematics itself. The word 'good' itself encapsulates several constraints: predictions and explanations must achieve a balance of being accurate, interesting, powerful, simple, and fruitful, and must be justifiable using two modes of reasoning: proof and interpretation. Finally sociological constraints may have some bearing (*e.g.*, mathematical beliefs need to further mathematicians in their careers and power struggles; the development of mathematics is no doubt constrained by the fact that the most popular conferences are in beach locations)—the question is how big a role such constraints play.

[3] See [Awodey, 2004] for a defence of a type of *if-then*-ism.

The objective Bayesian analogy then leads to an *epistemic* view of mathematics characterised by the following hypotheses:[4]

Convenience. Mathematical beliefs are convenient, because they admit good explanations and predictions within mathematics itself and also within its grounding interpretations.

Explanation. We have mathematical beliefs because of this convenience, not because uninterpreted mathematical entities correspond to physical things that we experience, nor because such entities correspond to platonic things that we somehow intuit.

Objectivity. The strength of the constraints on mathematical beliefs renders mathematics an objective, or nearly objective, activity.

Under the epistemic view, then, mathematics is like an axe. It is a tool whose design is largely determined by constraints placed on it.[5] Just as the design of an axe is roughly determined by its use (chopping wood) and demands on its strength and longevity, so too mathematics is roughly determined by its use (prediction and explanation) and a high standard of certainty as to its conclusions. No wonder that mathematicians working independently end up designing similar tools.

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[4] An analogous epistemic view of causality is developed in [Williamson, 2005a, Chapter 9].

[5] [Marquis, 1997, p. 252] discusses the claim that mathematics contains tools or instruments as well as an independent reality of uninterpreted mathematical entities. The epistemic position, however, is purely instrumentalist: there are tools but no independent reality. As Marquis notes, the former view has somehow to demarcate between mathematical objects and tools—by no means an easy task.